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Iterative Method for Nonlinear FM Synthesis of Radar Signals

The problem of synthesizing a time-domain signal with a given energy spectral density (ESD) often arises in the field of signal processing. Many solutions have been proposed and successfully used over the years. However the problem of synthesizing a time-domain signal with constraints for a given ESD has not been investigated sufficiently. We propose a solution to one such constraint where the amplitude of the complex-valued time-domain signal is required to be unity. This is equivalent to phase modulating a unit amplitude signal, such that its ESD matches a desired energy spectral density. We provide an algorithm for this solution and apply it to a real problem encountered in radars.

I. INTRODUCTION

In signal processing the energy spectral density (ESD) is the distribution of energy with frequency.

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For an M-point complex discrete-time signal $a(n)$, $n = 0,1,\ldots,M-1$, the N-point ESD $\mathcal{E}(k)$, $k = 0,1,\ldots,N-1$ is given by the square of the magnitude of the N-point discrete Fourier transform (DFT) $A(k), k = 0,1,\ldots,N-1$ of the discrete-time signal $a(n)$

$$A(k) = \sum_{n=0}^{M-1} a(n)\exp(-j2\pi kn/N), \qquad k = 0,1,\ldots,N-1 \tag{1}$$

as

$$\mathcal{E}(k) = |A(k)|^2.$$

If we have the ESD, it is possible to compute the discrete-time signal using the inverse Fourier transform and some window-function techniques, such as a rectangular window, a hanning window, a hamming window, a Blackman window, etc.

$$\tilde{a}(n) = \frac{1}{N}\sum_{k=0}^{N-1}\sqrt{\mathcal{E}(k)}\exp(j2\pi kn/N), \qquad n = 0,1,\ldots,N-1$$

$$a(n) = \tilde{a}(n)w(n)$$

where $\tilde{a}(n)$ is an N-point discrete-time signal and where $w(n)$ is an N-point window that reduces $\tilde{a}(n)$ to M non-zero points.

This problem of synthesizing a time-domain signal from its ESD often arises in the field of signal processing. For example when a signal has to be transmitted through a noisy medium, the optimal signal to transmit should have its energy in the noise bands with minimum power. In such cases the transmit signal may have to be synthesized to have a given ESD. This problem can be interpreted as designing a filter with a given ESD. Providing an impulse excitation to the filter generates the desired signal. Many techniques have been developed over the years to address this problem [1]. The computed filter coefficients are not always practically realizable for a variety of reasons. This quite often imposes constraints on the filter coefficients. Specific techniques need to be developed based on the particular constraints in the problem. One such constraint that has not been extensively investigated is the constraint for the amplitude of the filter coefficients. For example in radar the detection performance of the radar depends upon the ESD of the transmitted signal. A common practice is to use class C amplifiers, where the amplitude of the transmitted signal cannot be varied, and so, only phase or frequency modulation can be used. This is called angle modulation. When the modulation signal is nonlinear, the resulting signal is called a nonlinear frequency modulation (FM) signal [7]. In such a case if a transmit signal has to be synthesized for a given ESD, the only part of the signal that can be manipulated is the phase of the signal. The simplest way to do this would be to synthesize the signal without any constraint and then to force the amplitude of the signal to be unity as

$$a(n) = \mathcal{F}^{-1}\left\{\sqrt{\mathcal{E}_D(k)}\right\} \qquad \text{and} \qquad a_R(n) = \frac{a(n)}{|a(n)|}, \quad n = 0,1,\ldots,M-1$$

where $\mathcal{E}_D(k)$, $k = 0,1,\ldots,N-1$ are N-sample points of the desired ESD, $a_R(n)$ is the final synthesized signal, and where $\mathcal{F}^{-1}$ represents the inverse Fourier transform. However the ESD of the signal synthesized using this procedure may be drastically different from the given ESD. More sophisticated techniques are, therefore, needed to synthesize a signal that meets all the requirements. A standard technique commonly used for synthesizing a nonlinear FM signal is the stationary-phase technique [2]. The principle of stationary phase relies on the simple fact that most of the energy of a complex signal is concentrated around what are called stationary points. It can be explained in relation to the problem at hand as follows. Suppose that

$$u(t) = \exp[j\theta(t)]$$

is a complex unit amplitude frequency modulated signal with instantaneous frequency

$$\zeta(t) = \theta'(t).$$

The Fourier transform of this signal is given by

$$U(\omega) = \int_{-\infty}^{\infty} \exp[j(-\omega t + \theta(t))]dt.$$

The presence of an imaginary exponential makes the above function an oscillating function. A major contribution to the Fourier spectrum occurs when the rate of change of oscillation is minimal. The point (t_0,ω_0) that satisfies the condition

$$\frac{d}{dt}[\omega t - \theta(t)] = 0$$

is called the stationary point. Using the above result, and, with further assumption that $\zeta(t)$ is monotonically increasing, it is possible to find a function $\theta(t)$ such that $u(t)$ has the desired ESD. In Section III we provide the performance results of this technique as a comparison with our proposed technique. In [3] a technique is proposed whereby the error between the desired ESD and the ESD of the realized signal is minimized using the method of Lagrangian multipliers. This technique involves complex analytical derivations followed by computing a numerical solution using Newton's method. Some other methods are proposed in [4] and [5]. In this paper we propose an algorithm that is not necessarily optimal but is fairly simple, with less computations for real-time on-line processing, and it appears to produce a reasonably good solution.

II. PROBLEM STATEMENT AND PROPOSED SOLUTION

Given a desired ESD $\mathcal{E}_D(k)$, $k = 0,1,\ldots,N-1$, corresponding to N uniformly-spaced real DFT samples around the unit circle in the z-plane, we wish to approximate the desired magnitude spectrum $S(k)(= \sqrt{\mathcal{E}_D(k)})$ by the magnitude of the DFT $A(k)$ for an M-point unit-amplitude complex signal given by

$$a(n) = \exp(j\theta_n), \qquad n = 0,1,\ldots,M-1. \tag{2}$$

If $M < N$, $a(n)$ is zero-padded to N samples. Let $A(k)$ be the DFT of $a(n)$ given by (1). We wish to choose unit amplitude complex signal $a(n)$, such that the magnitude spectrum $|A(k)|$ of the signal is closest to the desired magnitude spectrum. This can be achieved by minimizing the following least squares error

$$J(\boldsymbol{\theta}) = \sum_{k=0}^{N-1}[S(k) - |A(k)|]^2$$

where $\boldsymbol{\theta} = [\theta_1\theta_2\cdots\theta_M]^T$. However to improve the approximation of the desired ESD, we associate the phase of $A(k)$ with the desired magnitude spectrum $S(k)$. If $\phi(k)$ is the phase of $A(k)$, then the least squares error can be rewritten as

$$J(\boldsymbol{\theta}) = \sum_{k=0}^{N-1}|S(k)e^{j\phi(k)} - A(k)|^2.$$

Let $\mathbf{x} = [S(0)e^{j\phi(0)}\ \ S(1)e^{j\phi(1)}\cdots S(N-1)e^{j\phi(N-1)}]^T$, $\mathbf{a} = [a(0)\ \ a(1)\cdots a(M-1)]^T$, and $\mathbf{A} = [A(0)\ \ A(1)\cdots A(N-1)]^T$. Let $\mathbf{W}$ be an $N \times M$ DFT matrix defined as

$$\mathbf{W} = \frac{1}{\sqrt{N}}\begin{bmatrix} \omega^{0\cdot 0} & \omega^{0\cdot 1} & \cdots & \omega^{0\cdot(M-1)} \\ \omega^{1\cdot 0} & \omega^{1\cdot 1} & \cdots & \omega^{1\cdot(M-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^{(N-1)\cdot 0} & \omega^{(N-1)\cdot 1} & \cdots & \omega^{(N-1)\cdot(M-1)} \end{bmatrix}$$

so that $\mathbf{A} = \mathbf{W}\mathbf{a}$ where $\omega = e^{-j(2\pi/N)}$. The least squares error can be expressed in vector form as

$$J(\boldsymbol{\theta}) = (\mathbf{x} - \mathbf{W}\mathbf{a})^H(\mathbf{x} - \mathbf{W}\mathbf{a})$$

where the exponent H stands for Hermitian (conjugate transpose). This problem can be interpreted as estimating the vector $\mathbf{a}$ based on observed values $\mathbf{x}$. Note that this is a nonlinear least squares problem because of the nonlinear dependence of $\mathbf{a}$ on $\boldsymbol{\theta}$ (see (2)). The linear least squares estimator of $\mathbf{a}$ is given by [6]

$$\hat{\mathbf{a}} = (\mathbf{W}^H\mathbf{W})^{-1}\mathbf{W}^H\mathbf{x}.$$

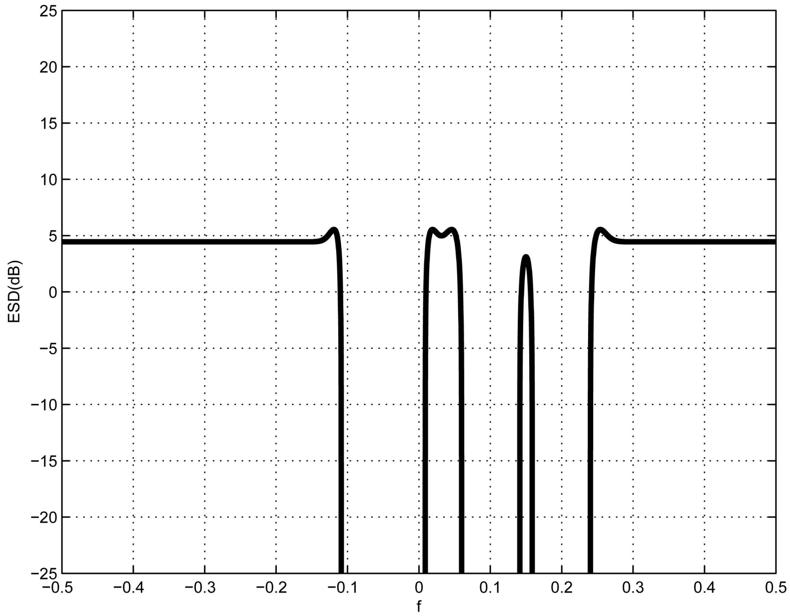

Fig. 1. Optimal transmit ESD computed in [8].

Since $\mathbf{W}$ is an $N \times M$ DFT matrix, which has orthogonal columns

$$(\mathbf{W}^H\mathbf{W})^{-1} = (1/N)\mathbf{I}_M$$

where $\mathbf{I}_M$ is the $M \times M$ identity matrix. The expression for the least squares estimator reduces to

$$\hat{\mathbf{a}} = (1/N)\mathbf{W}^H\mathbf{x}. \tag{3}$$

Since the vector $\mathbf{x}$ depends on the phase $\phi(k)$ of the DFT of the signal, which in turn depends on the signal, the solution must be solved for iteratively. To describe the ith stage of the iterative algorithm, let $\hat{\mathbf{a}}^{(i)}$ be the linear least squares estimate of the signal given by (3), let $\hat{\mathbf{A}}^{(i)}$ be the $N \times 1$ column vector of corresponding DFT values, and let $\mathbf{x}^{(i-1)}$ be the $N \times 1$ vector of the magnitude spectrum values $S(k)$ associated with the corresponding phase $\hat{\phi}^{(i-1)}(k)$, of $\hat{A}^{(i-1)}(k)$ from the previous iteration. The least squares estimate for the ith iteration is given by

$$\hat{\mathbf{a}}^{(i)} = (1/N)\mathbf{W}^H\mathbf{x}^{(i-1)} \tag{4}$$

where $\mathbf{x}^{(i-1)} = [S(0)e^{j\hat{\phi}^{(i-1)}(0)} \; S(1)e^{j\hat{\phi}^{(i-1)}(1)} \cdots S(N-1)e^{j\hat{\phi}^{(i-1)}(N-1)}]^T$. The criterion for convergence can be based on the norm of the variation, which is defined as

$$\Delta = \sum_{n=0}^{M-1} |\hat{a}^{(i)}(n) - \hat{a}^{(i-1)}(n)|$$

where $\hat{a}^{(i)}(n)$, $n = 0,1,\ldots,M-1$ is the least squares estimate in the ith iteration and where $\hat{a}^{(i-1)}(n)$, $n = 0,1,\ldots,M-1$ is the least squares estimate in the $(i-1)$th iteration.

The steps to compute the filter coefficients can be summarized as follows.

1) Let $\hat{\phi}^{(0)}(k) = 0$ for $k = 0,1,\ldots,N-1$. Compute $\mathbf{x}^{(0)} = [S(0)e^{j\hat{\phi}^{(0)}(0)} \; S(1)e^{j\hat{\phi}^{(0)}(1)} \cdots S(N-1)e^{j\hat{\phi}^{(0)}(N-1)}]^T$.

2) Compute $\hat{\mathbf{a}}^{(1)}$ using (4).

3) Compute $\bar{\hat{\mathbf{a}}}^{(1)}$, the filter coefficients with unit amplitude, from $\hat{\mathbf{a}}^{(1)}$ by dividing each coefficient by its amplitude to produce normalized coefficients.

4) Compute $\bar{\hat{\mathbf{A}}}^{(1)}$, the DFT of the normalized filter coefficients, using $\bar{\hat{\mathbf{A}}}^{(1)} = \mathbf{W}\bar{\hat{\mathbf{a}}}^{(1)}$.

5) Compute the phase $\hat{\phi}^{(1)}(k)$, of $\bar{\hat{\mathbf{A}}}^{(1)}$ for next iteration. Go to step 1.

III. SOME EXAMPLES

A. Synthesizing a Transmit Signal for a Radar from the Optimal Transmit ESD

Consider the problem of synthesizing an optimal transmit signal for a radar in the presence of noise and clutter. In [8] the ambient noise is modeled as a wide sense stationary (WSS) Gaussian random process with zero mean and power spectral density (PSD) $P_n(f)$. The clutter return is modeled as the output of a random linear time-invariant (LTI) filter with impulse response that is a complex WSS Gaussian random process, with zero mean and PSD $P_h(f)$. It is shown that the ESD of the optimal transmit signal is given by

$$\mathcal{E}_s(f) = \max\left(\frac{\sqrt{P_n(f)/\lambda} - P_n(f)}{P_h(f)}, 0\right)$$

where λ is a positive number. For a particular case the optimal ESD is computed at 5001 ($N = 5001$) frequency points, and it is plotted in Fig. 1.

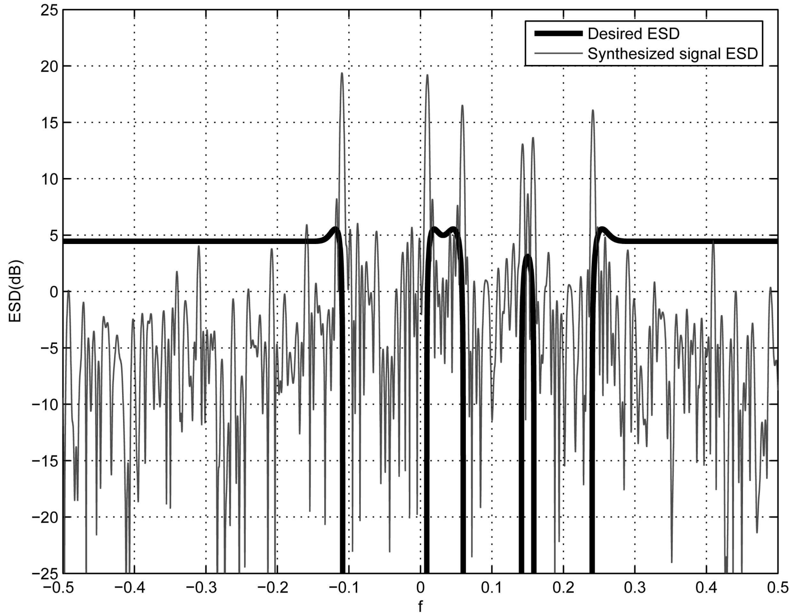

Fig. 2. Desired ESD and computed ESD for signal synthesized with simple technique with $M = 200$.

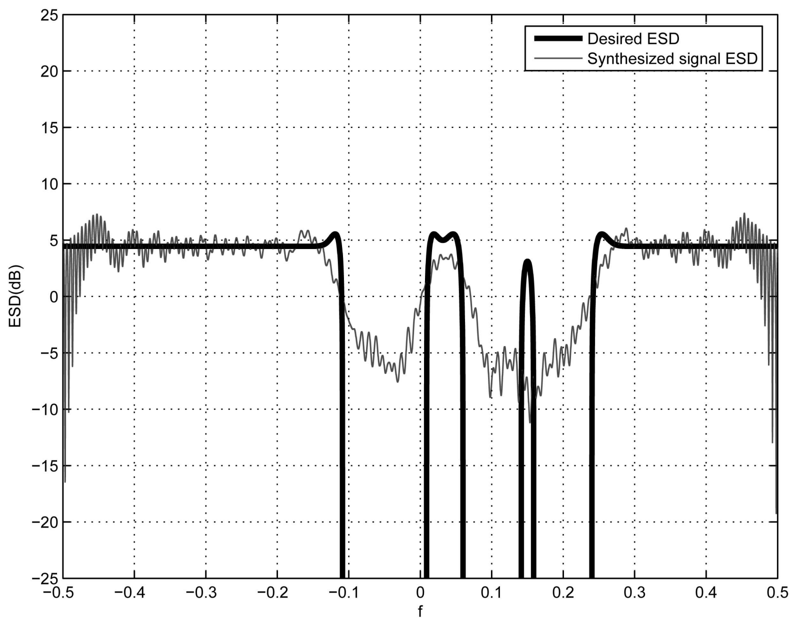

Fig. 3. Desired ESD and ESD of signal synthesized using stationary-phase technique with $M = 200$.

In the paper Durbin's method is used to synthesize the signal corresponding to this ESD. There is no constraint imposed on the coefficients of the signal. A simple way to synthesize a signal with this ESD, which satisfies the constraint of unit amplitude coefficients, is to compute the inverse DFT of the square root of the desired ESD, to window it to the required number of samples, and then to impose the constraint of unit amplitude on the resulting coefficients. Fig. 2 shows the ESD of a signal synthesized using this method, with $M = 200$, along with the desired ESD. The drastic difference between the ESD of the synthesized signal and the desired ESD can be clearly noticed. Fig. 3 has the ESD of a signal synthesized using the stationary-phase technique for the same value of $M = 200$.

We use the same ESD to generate a signal using our algorithm, with $M = 200$. The algorithm is iterated until $\Delta < 0.01 \times M (= 2)$, and it converges in 35 iterations. Additionally the algorithm is run for values of $M = 10, 50$ and 100, and it converges in 17, 18, and 25 iterations, respectively. Fig. 4 shows the computed ESD along with the desired ESD for $M = 200$. For a filter with filter coefficients designed using this algorithm, the average attenuation in the stopband is about 15.4523 dB below the average attenuation in the passband. This can be compared against the average attenuation difference between passband

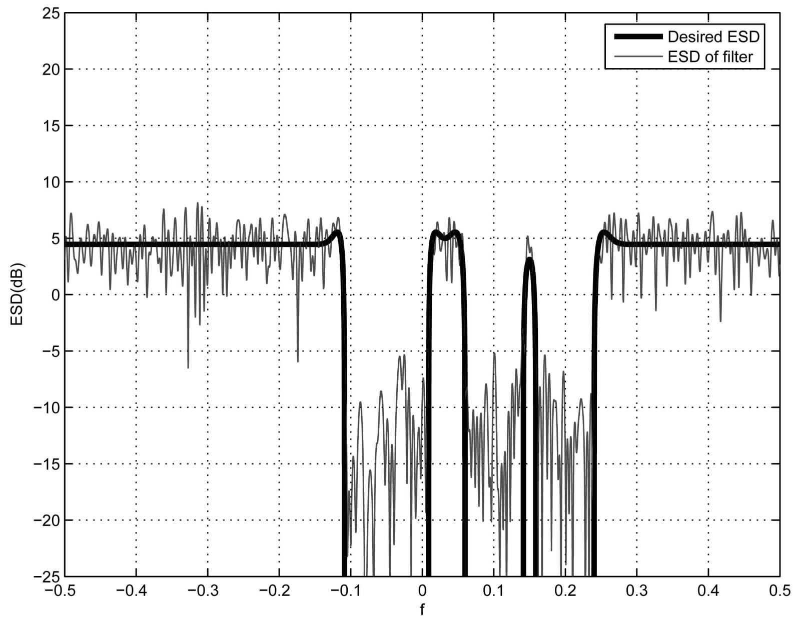

Fig. 4. Desired ESD and ESD of signal synthesized using proposed technique with $M = 200$.

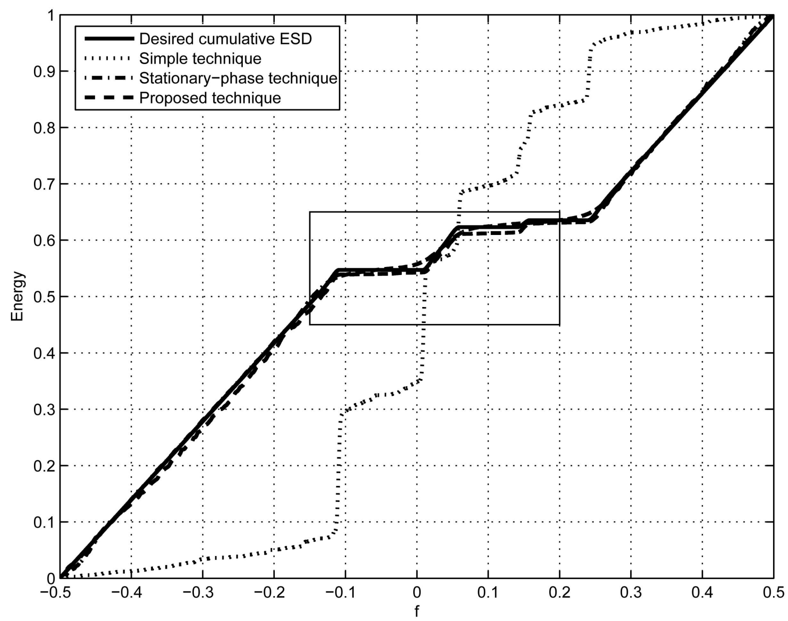

Fig. 5. Desired normalized cumulative ESD and normalized cumulative ESD of signal synthesized using simple, stationary-phase and proposed techniques with $M = 200$.

and stopband for the stationary-phase technique, which is 7.7731 dB. This is quite good for many practical applications. The closeness of the ESD of the signal synthesized using the proposed technique to the desired ESD can be visualized better by looking at the plots of the normalized cumulative ESDs in Fig. 5. This can be compared against the cumulative ESD of the signal synthesized using the simple and stationary-phase techniques. Normalized cumulative ESD is defined as

$$\hat{\mathcal{E}}(k) = \frac{\sum_{i=0}^{k} \mathcal{E}(i)}{\sum_{i=0}^{N-1} \mathcal{E}(i)}, \qquad k = 0,\ldots,N-1$$

Fig. 6 has the zoomed version of the section highlighted in a box in Fig. 5. It can be seen clearly in Fig. 6 that the stationary phase technique smooths out around the edges, while the proposed technique follows the desired cumulative ESD very closely, even at the edges. This means that when the desired ESD has a large number of sudden fluctuations, the proposed technique outperforms the stationary phase technique. The sections where the desired cumulative ESD is flat represent the stop band. The flatter the cumulative ESD, the higher the attenuation in the stopband. It is easy to notice that the proposed technique has much

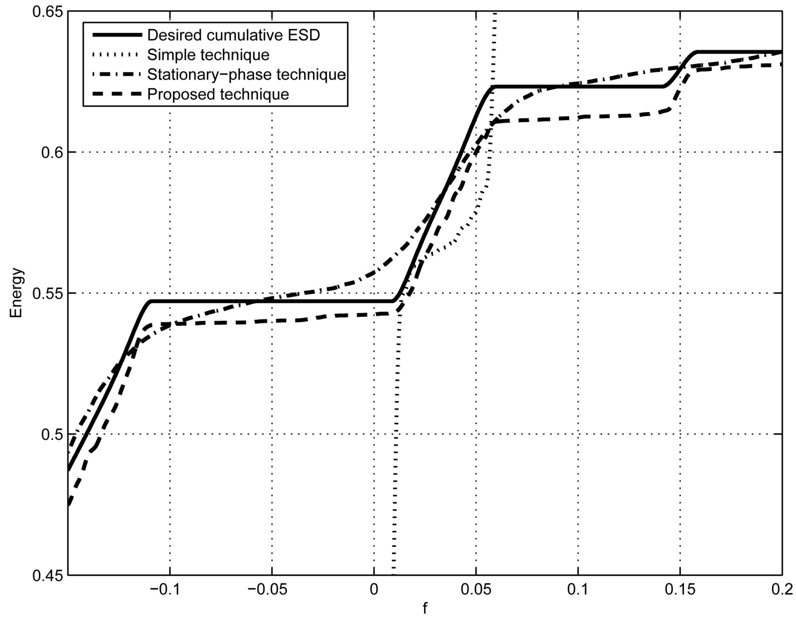

Fig. 6. Desired normalized cumulative ESD and normalized cumulative ESD of signal synthesized using simple, stationary-phase and proposed techniques with $M = 200$.

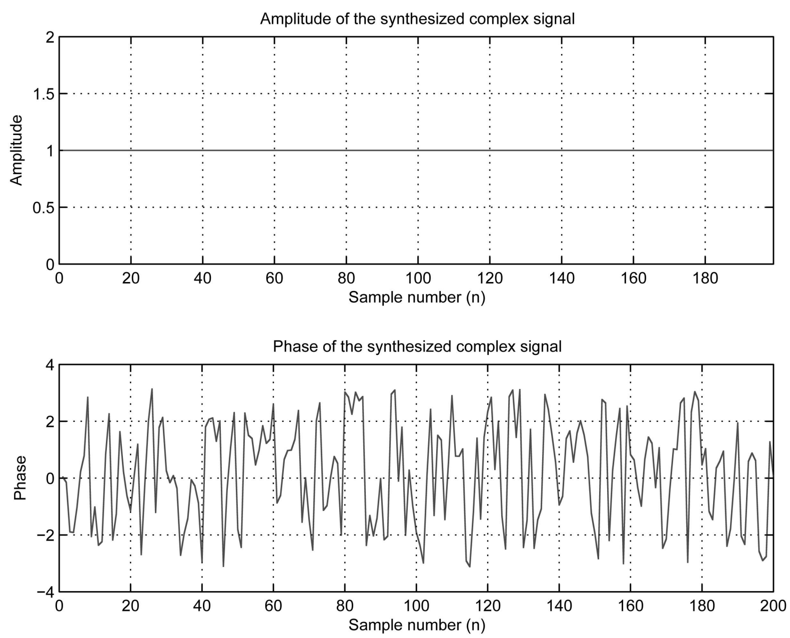

Fig. 7. Amplitude and phase of synthesized complex signal.

flatter cumulative ESD when compared with the stationary phase technique. A rudimentary comparison of the computation time revealed that all of this gain in performance is obtained with the proposed technique only at a cost of twice the computation time over the stationary phase technique. Fig. 7 has the amplitude and phase plots of the synthesized signal. It can be clearly seen that the amplitude of the synthesized signal is unity. Fig. 8 shows the norm of the variation of $\bar{\hat{\mathbf{a}}}$ between iterations as a function of the iteration number. It can be seen that this value decreases very rapidly, which implies that the algorithm converges to a solution very quickly.

Higher values of M provide more flexibility to the algorithm by increasing the degrees of freedom. However after the algorithm is provided with sufficient degrees of freedom to synthesize a certain ESD, increasing the value of M any further does not necessarily improve the performance of the algorithm. We ran the algorithm for different values of M. It was observed that the synthesized ESD gets closer to the desired ESD as M increases, as expected. But the improvement is not significant

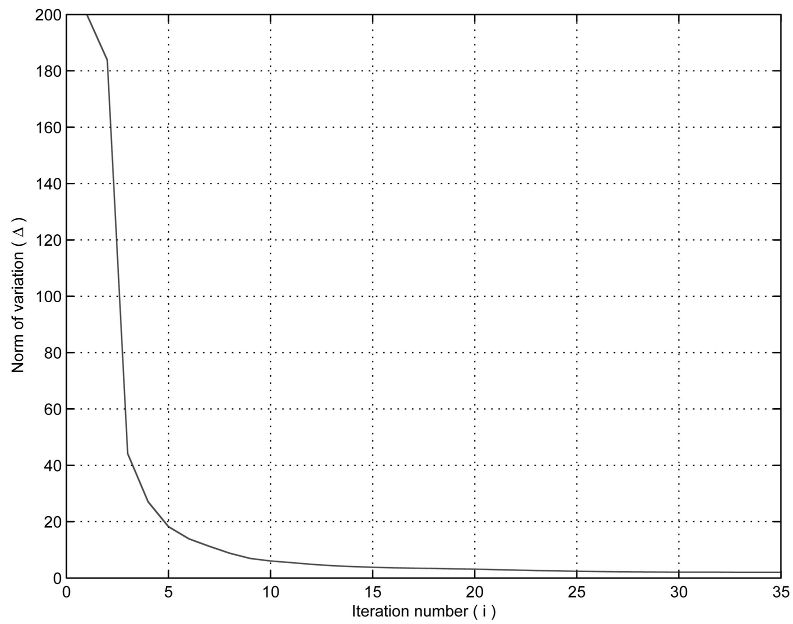

Fig. 8. Norm of variation of $\bar{\hat{\mathbf{a}}}$ between iterations.

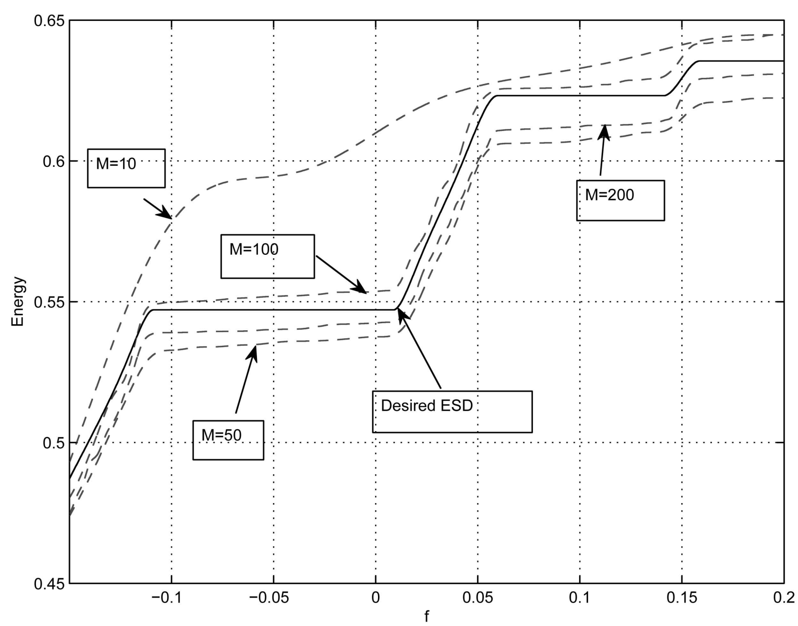

Fig. 9. Desired normalized cumulative ESD and normalized cumulative ESD of signal synthesized using proposed technique for different values of M.

after a certain value of M. This can be seen from the cumulative ESD plots in Fig. 9, where a section of the cumulative ESD of the desired signal is plotted along with the cumulative ESDs of the signals synthesized using the proposed technique for values of $M = 10$, 50, 100, and 200. The improvement is very prominent when M is increased from 10 to 50 and 50 to 100, but when M is increased from 100 to 200, we do not notice any further improvement. However it is important that M be less than N in order for the algorithm to converge to a good solution.

IV. CONCLUSIONS

We proposed a technique to synthesize a signal that has an ESD closest to a given ESD and that satisfies the condition that all the coefficients of the signal have unit amplitude. We give an iterative algorithm to solve this problem and show that this algorithm appears to converge to a reasonably good solution. An example is provided to show that the algorithm converges for a desired ESD used in radar for clutter suppression [8].

LELAND JACKSON
STEVEN KAY
NARESH VANKAYALAPATI
Dept. of Electrical, Computer, and Biomedical Engineering
University of Rhode Island
4 East Alumni Ave.
Kelley Hall, Room A121
Kingston, RI 02881
E-mail: (Jackson@ele.uri.edu)

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Far-Field Limit of PFA for SAR Moving Target Imaging

A new formulation of the polar format algorithm (PFA) for spotlight synthetic aperture radar (SAR) imaging is presented, which is useful when investigating its response to the moving target. Our work starts with examining the commonly adopted implementation of the PFA, i.e., the separable 1-D range and azimuth resampling procedures for their roles in range cell migration (RCM) correction, respectively. It is shown that the former performs RCM predeformation, while the latter is, substantially, the combination of RCM linearization and the keystone transform (KT), i.e., a well-known technique for SAR imaging of moving targets. The new formulation approach is applied to analyze the moving target. The far-field limit of the PFA for moving target focusing is derived, which can be used to predict the target's impulse response (IPR) function. The work presented might be helpful when considering a SAR system with the capability of ground moving target indication and imaging (GMTI&Im).

I. INTRODUCTION

The polar format algorithm (PFA) is well established in fine resolution spotlight synthetic aperture radar (SAR) processing. The classical theory of radar imaging formulates the algorithm by indicating an intrinsic but simple Fourier transform relationship between the complex reflectivity of the illuminated scene and the collected data [4–7]. In practice the 2-D resampling of the data samples is prerequisite to exploiting the efficiency of the fast Fourier transform (FFT), and the commonly adopted implementation of which is the separable 1-D range and azimuth resampling.

The derivation of the PFA can be found in [4]–[7], and [11]. They are adequate to clarify the 2-D Fourier transform relationship and, meanwhile, to derive the far-field limit. Nevertheless the mechanism of range cell migration (RCM) correction in the PFA has not been illuminated explicitly in these

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